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Golden Proportion for Anexclusive Class of generalized Fibonacci Numbers

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ABSTRACT: There is an interesting combination of Fibonacci and Lucas sequences in the form of sequence $\{K_n\}$ defined by the recurrence relation $K_n + K_{n+1} = K_{n+2}; n \geq 0$ with the initial conditions $K_0 = 2m, K_1 = k + m$; where m, k are fixed positive integers. Here the initial conditions are the sum of 'k' times the initial conditions of traditional Fibonacci sequence and 'm' times the initial conditions of traditional Lucas sequence. We further generalize this sequence defined by the recurrence relation $G_n + G_{n+p} = G_{n+q}$ with the initial conditions $G_0 = 2m, G_1 = k + m$ and m, k, p, q are fixed positive integers with $1 \leq p < q$. In this paper, we obtain the 'generalized golden proportion' for this new family of generalized Fibonacci numbers and show that this proportion is independent of the values of m and k . In fact, we prove that when both p, q are not simultaneously even, $\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} = M^t$; for some real number M and $t = 1, 2, 3, \dots$

KEYWORDS: Fibonacci sequence, Generalized Fibonacci sequence, Golden proportion.

I. INTRODUCTION

The sequence $\{F_n\}$ of Fibonacci numbers is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 2$ with initial condition $F_0 = 0, F_1 = 1$. Also the sequence of Lucas numbers is defined by $L_n = L_{n-1} + L_{n-2}$, for all $n \geq 2$ with initial condition $L_0 = 2, L_1 = 1$. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions and others by preserving the recurrence relation. One can refer [1, 3, 5, 8, 9, 11, 12] for various generalized Fibonacci sequences. Badshah, Teeth and Dar [2] considered an interesting combination of Fibonacci and Lucas sequences in the form $M_n = M_{n-1} + M_{n-2}; n \geq 2$ with $M_0 = 2m, M_1 = 1 + m$; where m is fixed positive integer. Here the initial conditions are the sum of the initial conditions of traditional Fibonacci sequence and 'm' times the initial conditions of traditional Lucas sequence. Patel and Shah [10] considered the analogues problem and made further generalization of this sequence. They defined the sequence $\{K_n\}$ as under, where the initial conditions are the sum of 'k' times the initial conditions of traditional Fibonacci sequence and 'm' times the initial conditions of traditional Lucas sequence.

Definition: The terms of the sequence $\{K_n\}$ are defined by the recurrence relation $K_n = K_{n-1} + K_{n-2}; n \geq 2$ with the initial conditions $K_0 = 2m, K_1 = k + m$; where m, k are fixed positive integers.

First few terms of this sequence are $2m, k + m, k + 3m, 2k + 4m, 3k + 7m, 5k + 11m, \dots$

Moreover, it is well known that the ratios of consecutive terms of Fibonacci sequence converges to the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ which is the positive root of the equation $x^2 - x - 1 = 0$ (See: Vajda[14]). Stakhov[13] defined the p -Fibonacci numbers $F_p(n)$ for $p \geq 1$ by the recurrence relation

$$F_p(n) = \begin{cases} 1 & ; \text{for } 1 \leq n \leq p + 1 \\ F_p(n-1) + F_p(n-p-1) & ; \text{for } n > p + 1 \end{cases}$$

It can be seen that by taking $p = 1$, the recurrence relation becomes $F(n) = F(n-1) + F(n-2); n > 2$; where $F(1) = F(2) = 1$. This is a well-known Fibonacci sequence.

The values of $F_p(n)$ for $p = 1, 2, \dots, 6$ and for first 15 values of n are shown below for the ready reference.

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n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$F_1(n)$	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$F_2(n)$	1	1	1	2	3	4	6	9	13	19	28	41	60	88	129
$F_3(n)$	1	1	1	1	2	3	4	5	7	10	14	19	26	36	50
$F_4(n)$	1	1	1	1	1	2	3	4	5	6	8	11	15	20	26
$F_5(n)$	1	1	1	1	1	1	2	3	4	5	6	7	9	12	16
$F_6(n)$	1	1	1	1	1	1	1	2	3	4	5	6	7	8	10

Values of $F_p(n)$

Stakhov[13] also shown that $F_p(n)$ satisfies $\lim_{n \rightarrow \infty} \frac{F_p(n)}{F_p(n-1)} = \phi_p$, where the ‘golden p -proportion’ ϕ_p is the root of $x^{p+1} = x^p + 1$. De Villiers [4] made the similar observation and provided the partial proof of the above result containing limiting ratio in the case when p is odd, with the suggestions for the case when p is even. Later Falcon [7] generalized the same problem and provided the complete proof for the same.

Here we consider the analogous problem for the further generalization of $\{K_n\}$ and consider the problem of finding the ‘golden proportion’ for the whole family of this new generalized sequence.

II. PRELIMINARIES AND ANALYSIS OF THE ROOTS

We now consider the further generalization of sequence $\{K_n\}$ whose terms are defined by the recurrence relation

$$G_n + G_{n+p} = G_{n+q}; \tag{2.1}$$

with the initial conditions $G_0 = 2m, G_1 = k + m$ and m, k, p, q are fixed positive integers with $1 \leq p < q$. We note that if we consider $p = 1, q = 2, m = 0, k = 1$ then $G_n = F_n$. Also if we choose $p = 1, q = 2, m = 1, k = 0$ then $G_n = L_n$. In this paper, we obtain the ‘generalized golden proportion’ for this new family $\{G_n\}$ of generalized Fibonacci numbers and show that this proportion is independent of the choice of m and k .

Since $q > p$, we let $q = p + a$. Then difference equation (2.1) becomes $G_{n+p+a} = G_n + G_{n+p}$. By substituting n for $n + p + a$, this difference equation can be written as $G_n = G_{n-a-p} + G_{n-a}$. The corresponding characteristic equation for this difference equation is $x^n = x^{n-a-p} + x^{n-a}$. Now dividing this equation by x^{n-a-p} , we have $x^{n-n+a+p} = 1 + x^{n-a-n+a+p}$. This on simplification takes the form $x^p(x^a - 1) = 1$. This is equivalent to

$$x^p = \frac{1}{x^a - 1}. \tag{2.2}$$

We let $f(x) = x^p$ and $g(x) = \frac{1}{x^a - 1}$.

Now we observe that $f(x) = x^p$ is a parabolic type curve, if p is even. In this case the intersection of two curves $f(x)$ and $g(x)$ is shown in figures 1 and 2.

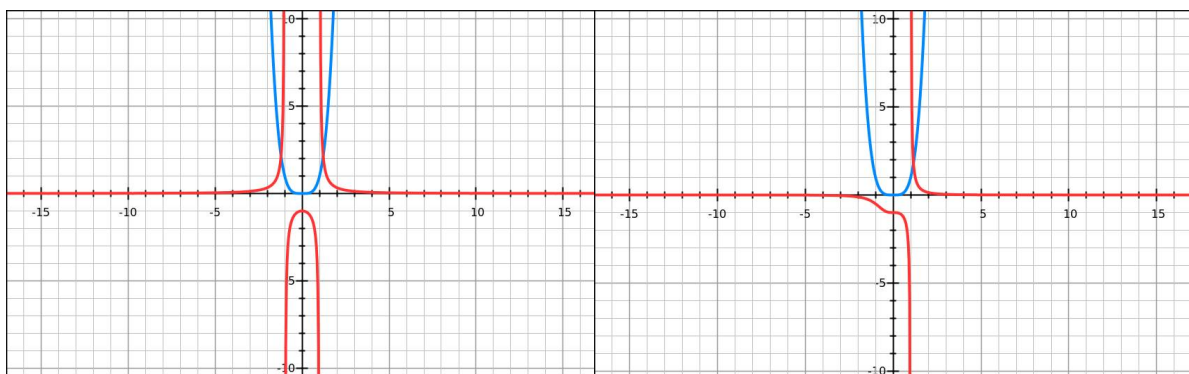


Figure 1 (p even, a even) Figure 2 (p even, a odd)

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Also, if p is odd, the graph of $f(x) = x^p$ is a curve of cubic type. In this case the intersection of two curves $f(x)$ and $g(x)$ is shown in figures 3 and 4.

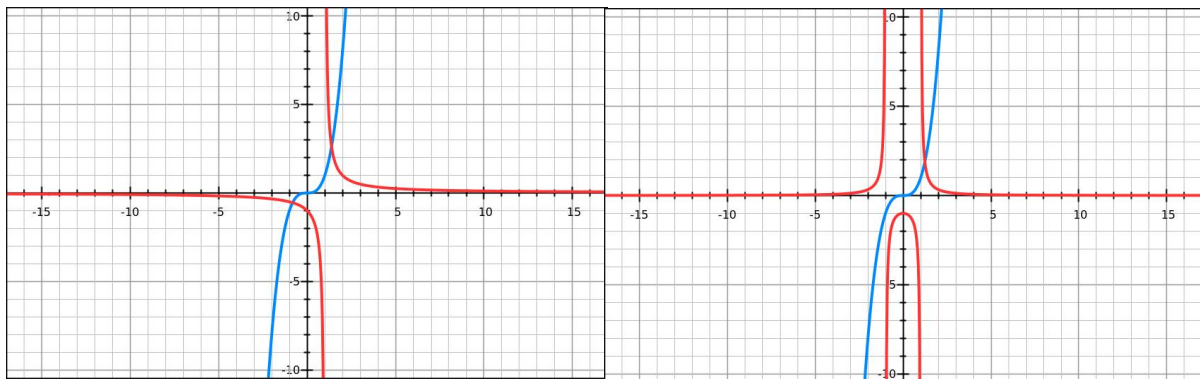


Figure 3(p odd, a odd) Figure 4 (p odd, a even)

It can be now noted that when p and a are even, system (2.2) has two real roots $x = \pm M$ where $1 < M < 2$; whereas when both p and a are odd, it has two real roots $x = M$ and $x = -M_1$ for $0 < M_1 < 1 < M < 2$. Also when one of p and a is even and other is odd, system has one real root $x = M$ for some M such that $1 < M < 2$.

In short: The roots of (2.2) are

- (a) One positive real number M , where $1 < M < 2$.
- (b) One real number $-M$, where $1 < M < 2$; when both p, a are even.
- (c) One real number $-M_1$, where $0 < M_1 < 1$; when both p, a are odd.

At this point we note that total number of roots of (2.2) are $p + a$. The roots of (2.2) other than mentioned above are simple complex numbers Z whose modulus is always less than M . We express these complex roots in exponent form as $Z_j = r_j e^{i\theta_j}$; where $r_j = \sqrt{a_j^2 + b_j^2}$, $\theta_j = \tan^{-1} \left(\frac{b_j}{a_j} \right)$ and also $r_j < 1$, for all j .

III. THE GENERALIZED RATIO FOR THE GENERALIZED FIBONACCI NUMBERS

Before we prove the main result, we need to prove some intermediate results which when combined will give the main result. Throughout we consider t to be any fixed positive integer, a and b are any positive integers and $-1 < M_1 < 0 < M < 1$.

Lemma 1: When any one of p or a is even and other is odd,

$$\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} = M^t ; \text{where } = 1, 2, 3 \dots$$

Proof: In this case the characteristic equation (2.2) has only one real root. Taking into account the above discussion, by the theory of equations (Elaydi [6]) we can say that solution of the difference equation (2.2) is $G_n = a_1 M^n + \sum_{j=2}^{p+a} a_j Z_j^n$; where a_j 's are arbitrary constants.

$$\begin{aligned} \text{Thus, } \lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} &= \lim_{n \rightarrow \infty} \frac{G_n}{G_{n-t}} = \lim_{n \rightarrow \infty} \frac{a_1 M^n + \sum_{j=2}^{p+a} a_j Z_j^n}{a_1 M^{n-t} + \sum_{j=2}^{p+a} a_j Z_j^{n-t}} \\ &= \lim_{n \rightarrow \infty} \frac{a_1 + \sum_{j=2}^{p+a} a_j \left(\frac{Z_j}{M} \right)^n}{a_1 \left(\frac{1}{M^t} \right) + \sum_{j=2}^{p+a} a_j \left(\frac{Z_j}{M} \right)^{n-t} \left(\frac{1}{M^t} \right)} \end{aligned}$$

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$$= \lim_{n \rightarrow \infty} \frac{a_1 + \sum_{j=2}^{p+a} a_j \left(\frac{r_j}{M}\right)^n e^{in\theta_j}}{a_1 \left(\frac{1}{M^t}\right) + \sum_{j=2}^{p+a} a_j \left(\frac{r_j}{M}\right)^{n-t} \left(\frac{1}{M^t}\right) e^{i(n-t)\theta_j}}$$

Since $r_j < 1 < M$, for all $j = 2, 3, \dots, p + a$, we have

$$\lim_{n \rightarrow \infty} \left(\frac{r_j}{M}\right)^n = 0, \quad \lim_{n \rightarrow \infty} \left(\frac{r_j}{M}\right)^{n-t} = 0.$$

Thus $\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} = \frac{a_1}{a_1 \left(\frac{1}{M^t}\right)} = M^t$, which proves the result.

Lemma 2: When both p and a are even, $\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}}$ does not exist.

Proof: In this case it is seen that the characteristic equation (2.2) has two real roots $\pm M$.

Thus we write the solution of (4.2.3) as

$$G_n = a_1 M^n + a_2 (-M)^n + \sum_{j=3}^{p+a} a_j Z_j^n.$$

Now, $\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} = \lim_{n \rightarrow \infty} \frac{G_n}{G_{n-t}}$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{a_1 M^n + a_2 (-M)^n + \sum_{j=3}^{p+a} a_j Z_j^n}{a_1 M^{n-t} + a_2 (-M)^{n-t} + \sum_{j=3}^{p+a} a_j Z_j^{n-t}} \\ &= \lim_{n \rightarrow \infty} \frac{a_1 + a_2 (-1)^n + \sum_{j=3}^{p+a} a_j \left(\frac{Z_j}{M}\right)^n}{a_1 \left(\frac{1}{M^t}\right) + a_2 (-1)^n \left(\frac{1}{M}\right)^t + \sum_{j=3}^{p+a} a_j \left(\frac{Z_j}{M}\right)^{n-t} \left(\frac{1}{M^t}\right)} \\ &= \lim_{n \rightarrow \infty} \frac{a_1 + a_2 (-1)^n + \sum_{j=3}^{p+a} a_j \left(\frac{r_j}{M}\right)^n e^{in\theta_j}}{\left(\frac{1}{M^t}\right) \{a_1 + a_2 (-1)^{n-t}\} + \sum_{j=3}^{p+a} a_j \left(\frac{r_j}{M}\right)^{n-t} \left(\frac{1}{M^t}\right) e^{i(n-t)\theta_j}} \end{aligned}$$

This gives

$$\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} = \lim_{n \rightarrow \infty} \frac{a_1 + a_2 (-1)^n}{\left(\frac{1}{M^t}\right) \{a_1 + a_2 (-1)^{n-t}\}}.$$

It can be seen easily that this limit does not exist. This proves the required result.

Lemma 3: When both p and a are odd, $\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} = M^t$.

Proof: In this case it is seen that the equation (2.2) has two real roots M and $-M_1$, where $0 < M_1 < 1 < M < 2$.

Here we write the solution of (2.2) as $G_n = a_1 M^n + a_2 (-M_1)^n + \sum_{j=3}^{p+a} a_j Z_j^n$.

Now, $\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} = \lim_{n \rightarrow \infty} \frac{G_n}{G_{n-t}}$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{a_1 M^n + a_2 (-M_1)^n + \sum_{j=3}^{p+a} a_j Z_j^n}{a_1 M^{n-t} + a_2 (-M_1)^{n-t} + \sum_{j=3}^{p+a} a_j Z_j^{n-t}} \\ &= \lim_{n \rightarrow \infty} \frac{a_1 + a_2 \left(\frac{-M_1}{M}\right)^n + \sum_{j=3}^{p+a} a_j \left(\frac{r_j}{M}\right)^n e^{in\theta_j}}{a_1 \left(\frac{1}{M^t}\right) + a_2 \left(\frac{-M_1}{M}\right)^{n-t} \left(\frac{1}{M^t}\right) + \sum_{j=3}^{p+a} a_j \left(\frac{r_j}{M}\right)^{n-t} \left(\frac{1}{M^t}\right) e^{i(n-t)\theta_j}} \end{aligned}$$

Since $1 < M < 2$ and $0 < M_1 < 1$, using the facts that

$$\lim_{n \rightarrow \infty} \left(\frac{-M_1}{M}\right)^n = 0,$$

we get $\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} = \frac{a_1}{a_1 \left(\frac{1}{M^t}\right)} = M^t$, as required.

Combining all these lemma, we finally state the following theorem for recurrence relation (2.2).

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Theorem 4: For the sequence $\{G_n\}$ it is always true that

$$\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} = M^t;$$

unless both p and a are even.

We finally give the main result which follows from above theorem by considering $q = p + a$ in (2.1), where $p < q$ are integers.

Theorem 5: For the generalized sequence $\{G_n\}$ defined by $G_n + G_{n+p} = G_{n+q}$, it is always true that

$$\lim_{n \rightarrow \infty} \frac{G_{n+p+t}}{G_{n+p}} = M^t;$$

unless both p, q are even together.

This shows that the generalized ratio of generalized Fibonacci numbers is independent of the value of k and m .

REFERENCES

- [1] Arvadia, M. P., Shah D. V., "Right k-Fibonacci sequence and related identities", International Research Journal of Mathematics, Engineering and IT, Vol. 2, Issue-4, April 2015, 25 – 39.
- [2] Badshah V. H., Teeth M. S., Dar M. M., "Generalized Fibonacci-Like sequence and its properties", Int. J. Contemp. Math. Sciences, Vol.7, No.24, 2012, 1155 – 1164.
- [3] Das Khushbu J., Shah Devbhadra V., "A new approach to the generalized Lucas sequence" Mathematics Today, Vol. 32, June & December, 2016, 33 – 40.
- [4] De Villiers. M., "A Fibonacci generalization and its dual", Int. Jr. Math. Educ. Sci. Technol., Vol. 31, 2000, 464 – 473.
- [5] Diwan Daksha M., Shah Devbhadra V., "Some interesting properties and extended Binet formula for the Generalized Lucas sequence", International Journal of Innovative research in Science, Engineering and Technology, Vol. 4, Issue 12, Dec. 2015, 12832 – 12837.
- [6] Elaydi S. N., "An introduction to difference equations", Springer-Verlag, New York, 2ndEdn., 1999.
- [7] Falcon S., "A simple proof of an interesting Fibonacci generalization", Int. Jr. Math. Edu. Sci. Tech., Vol. 35, No. 2, 2004, 259 – 261.
- [8] Horadam A. F., "Basic properties of certain generalized sequence of numbers", The Fibonacci Quarterly, 3, 1965, 161–176.
- [9] Marcia Edson, Omer Yayenie, "A new generalization of Fibonacci sequence and extended Binet's formula", Integers, Volume 9, Issue 6, (2009), 639–654.
- [10] Patel Vandana R., Shah Devbhadra V., "An Interesting generalization of Fibonacci and Lucas sequence", International Journal of Science and Research, Vol. 4, Issue 12, Dec. 2015, 1942 – 1953.
- [11] Patel Vandana R., Shah Devbhadra V., "Generalized Fibonacci Sequence and its properties", International Journal of Physics and Mathematics Sciences, Vol. 4 (2), April-June (2014), 118–124.
- [12] Shah Mansi S., Shah D. V., "A new class of Generalized Lucas sequence", International Journal of Advanced Research in Engineering, Science and Management, April 2015, 1 – 7.
- [13] Stakhov A., "The golden section and modern harmony mathematics", Applications of Fibonacci numbers, Vol. 7, Kluwer Academic Pub., 1998, 393 – 399.
- [14] Vajda S., "Fibonacci and Lucas Numbers and the Golden Section: Theory and Applications", John Wiley and Sons, New York, 1989.